On the Regular and Normal Number of Topological Spaces and Cardinal Invariants

Maddalena Bonanzinga[1] Dimitrina Stavrova[2] Petra Staynova[3]

[1]Department of Mathematics
University of Messina

[2]Department of Mathematics
University of Leicester

[3]Department of Mathematics
Durham University

2014 International Conference on Topology and its Applications
3-7 July, Nafpaktos, Greece

Introduction

All notions not defined here can be found in [Juh80].

Definition ([Juh80])

A Hausdorff space X is called *strongly Hausdorff* iff from every infinite subset $A \subset X$ we can choose a sequence $\{p_n : n \in \omega\}$ such that p_n have pairwise disjoint neighbourhoods in X.

- 'Non-Hausdorff topologies and Domain Theory', Cambridge University Press 2013 [GL13]
- The Hausdorff number of a topological space has been introduced and studied in M. Bonanzinga's paper, submitted in 2011 to the Houston Journal of Mathematics [Bon13]
- Further observations of the Hausdorff number of finite and compact spaces have been made in [Sta13].

Introduction

Here we introduce two more generalisations of the standard separation axioms

- *n*-regular, and
- *n*-normal,

as well as two cardinal numbers related to those axioms

- the regular number $R(X)$, and
- the normal number $N(X)$ of a given topological space.

We will give several examples showing how known properties relate in such spaces, as well as (if time permits) new theorems about cardinal invariants. Open questions will be posed.

Let X be a topological space.

Definition ([Bon13])

The *Hausdorff number* of X, $H(X)$, is defined as:

$$H(X) = \min\{\tau : \text{ whenever } \{x_\alpha : \alpha \in \tau\} \text{ is a subset of different points in } X, \text{ then } \forall \alpha \in \tau \text{ there is an open } U_\alpha \subset X \text{ such that } x_\alpha \in U_\alpha \text{ and } \cap_{\alpha\in\tau} U_\alpha = \emptyset\}.$$

Definition

The *regular number* of X, $R(X)$, is defined as:

$$R(X) = \min\{\tau : \text{ whenever } F \subset X \text{ is closed and nonempty and } \{x_\alpha : \alpha \in \tau\} \text{ is a subset of different points in } X \text{ such that } F \cap \{x_\alpha : \alpha \in \tau\} = \emptyset \text{ then there are open sets } U \supset F, U_\alpha \subset X, \forall \alpha \in \tau, x_\alpha \in U_\alpha \text{ such that } U \cap \bigcap_{\alpha\in\tau} U_\alpha = \emptyset\}.$$

Definition

The *normal number* of X, $N(X)$, is defined as:

$$N(X) = \min\{\tau : \text{ whenever } \{F_\alpha : \alpha \in \tau\} \text{ is a disjoint family of closed nonempty different subsets of } X \text{ then there are open sets } U_\alpha \supset F, \ \forall \alpha \in \tau, \text{ such that } \cap_{\alpha \in \tau} U_\alpha = \emptyset\}.$$

If $H(X) \leqslant \tau$ ($R(X) \leqslant \tau$, $N(X) \leqslant \tau$) we call (respectively) X τ-Hausdorff (τ-regular, τ-normal).

When X is τ-Hausdorff (τ-regular, τ-normal) then it is β-Hausdorff (β-regular, β-normal) for every $\beta \geqslant \tau$.

In T_1 spaces we also have

$$H(X) \leqslant R(X) \leqslant N(X).$$

First Example

Example

There is a T_1, first-countable, compact, not normal, 3-normal space X with cardinality 2^ω.

Hence, even the strongest "non-Hausdorff" property does not imply Hausdorff (regular, normal) even in compact first-countable spaces.

Construction:
Define $X = ([0,1] \times \{0\}) \cup \{\langle \frac{1}{2}, \frac{1}{2} \rangle\}$. Topologize X as follows:

- all points on $[0,1] \times \{0\}$ have the Euclidean neighborhoods;
- the neighborhoods of $\langle \frac{1}{2}, \frac{1}{2} \rangle$ consist of

$$U_n\left(\left\langle \frac{1}{2}, \frac{1}{2} \right\rangle\right) = \left\{\left\langle \frac{1}{2}, \frac{1}{2} \right\rangle\right\} \cup \left(\left(\left(\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}\right) \setminus \left\{\frac{1}{2}\right\}\right) \times \{0\}\right).$$

Further examples

A similar idea can be used to construct:

Example

There is a T_1, first countable, compact space X with cardinality 2^ω such that X is $(n+1)$-normal but not n-normal for all $n \in \mathbb{N}$.

and also:

Example

There is an ω_1-normal space X which is first countable, Lindelöf, T_1, not n-normal (or n-regular, or n-Hausdorff) for any finite $n \geqslant 2$ and is not even ω-normal.

This is a slightly more complicated version of our first example. We can also obtain not-T_1-versions of the above spaces.

Theorem ([Gry80])

Every T_1 compact space with countable pseudocharacter has cardinality at most 2^ω.

We can modify our first example by not removing $\left\langle \frac{1}{2}, 0 \right\rangle$ from the neighbourhoods of $\left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$ in order to obtain

Example

There is a 3-normal not normal, not T_1 compact first countable space X with $|X| = 2^\omega$.

We no longer can talk about pseudocharacter in this space, but we can relate this example to Gryzlov's theorem via an open problem.

Open Problem

In order to formulate our first open question, let us recall a notion introduced in [Bon13]:

Definition ([Bon13])

Let X be n-Hausdorff space. Let the n-Hausdorff pseudocharacter be defined as:

$$n\text{-}H\psi(X) = \min\{\tau : \forall x \in X \text{ there is a collection of open neighbourhoods of } x, \{V(\alpha, x) : \alpha \in \tau\} \text{ such that for any } n \text{ distinct points } x_1, \dots, x_n \in X \text{ there are } \alpha_1, \dots, \alpha_n \in \tau \text{ such that } \cap_{i=1}^{n} V(\alpha_i, x_i) = \emptyset\}.$$

Open Question

Let X be 3-Hausdorff compact space with countable 3-Hausdorff pseudocharacter. Is it true that $|X| \leqslant 2^\omega$?

More Examples, Propositions, and Open Questions

We can construct examples for other infinite values of τ, as well, which possess better separation properties.

Example

There is a Hausdorff, regular, not 2^ω-normal space X with $|X| = 2^\omega$.

The Niemitzky plane is such an example.

One can speculate that the product of two n-normal spaces is $(n+k)$-normal for some finite $k \geqslant 1$.

But, for the above example, we can also take the Sorgenfrey plane. We then see that n-normality is not productive in a very bad way - even in the case of Hausdorff normal spaces (so the above speculation should be considered only for $n > 1$).

More Examples, Propositions, and Open Questions, cont'd

Example

There is a Hausdorff not ω-regular but ω_1-regular space X.

Take the real line $\mathbb{R}$ with the following topology τ: the basis of τ consists of all open intervals and all sets of the form $(a, b) \setminus K$, where $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. Any "tail" of K is closed and does not contain 0, and 0 and the "tail" do not have disjoint open neighbourhoods.
Let us also mention two more examples

Example

There is a Hausdorff countable 3-normal not normal space.

Instead of $\mathbb{R}$, we take $\mathbb{Q}$, and the same topology as previously described.
The idea can be used to construct:

Example

There is a Hausdorff countable $(n+1)$-normal not n-normal space.

We can prove various analogues of results relating separation properties in compact and other spaces. As an example

Proposition

If X is T_1 compact and $(n+1)$-Hausdorff space, then X is n-regular.

and

Proposition

If X is T_1 compact and $(n+1)$-Hausdorff space then it is $(n+1)$-normal.

Inheritance

It is well-known that the Hausdorff property is hereditary. It is clear that every subspace of a τ-Hausdorff space is τ-Hausdorff. However, some subspaces of n-Hausdorff spaces can have a Hausdorff number less than n.

Hence, it is interesting to make the following observations about some of the uncountable subspaces in the examples before:

- The very first example has an uncountable (of cardinality 2^ω) open compact subspace which is Hausdorff, namely $[0,1] \times \{0\}$.
- Analogously, we can have an uncountable open compact subspace which is $(n+1)$-Hausdorff but not n-Hausdorff.
- Moreover, the last two examples which we considered show that there is a space which is ω_1-regular, not ω-regular, with a subspace that is 3-normal.

Inheritance, cont'd

Hence, the Hausdorff number of a subspace is bounded above by the Hausdorff number of the whole space; we can conjecture that the same is true for the regular and normal numbers of a space.

This leads us to the following natural question:

Open Question

Given an uncountable τ-Hausdorff space X, do we have, for each $\kappa < \tau$, an uncountable κ-Hausdorff subspace of X?

Restriction of Cardinality

In the printed abstract there is a small misprint and the last theorem should read:

Theorem

If X is T_1 space then

$$|X| \leqslant \psi(X)^{hL(X)},$$

where $\psi(X)$ is the pseudocharacter and $hL(X)$ - the hereditarily Lindelöf number of X.

This is a slight improvement of a theorem of de Groot that in Hausdorff spaces, $|X| \leqslant 2^{hL(X)}$.

Restriction of Cardinality, cont'd

We can consider a cardinal invariant $3\text{-}\psi_c(X)$ in 3-Hausdorff spaces to give an analogue of the previous result.

Definition

Let X be 3-Hausdorff space and let

$$3\text{-}\psi_c(X) = \min\Big\{\tau : \forall x \in X \text{ there is a family } \mathcal{U}(x) \text{ of open subsets of } X \text{ such that } x \in \cap\mathcal{U}(x), |\mathcal{U}(x)| < \tau, \text{ and if } x, y, z \text{ are three distinct points, then either } y \notin \cap\{\overline{U} : U \in \mathcal{U}(x)\} \text{ or } z \notin \cap\{\overline{U} : U \in \mathcal{U}(x)\}\Big\}$$

Let us point out that in Hausdorff spaces,

$$3\text{-}\psi_c(X) = \psi_c(X) \leqslant hL(X).$$

Restriction of Cardinality, cont'd

We have the following result:

Theorem

Let X be a 3-Hausdorff space. Then

$$|X| \leqslant (3\text{-}\psi_c(X))^{haL(X)}.$$

Here, $haL(X)$ is the hereditarily almost Lindelöf number.

Thank you!

Thank you!

Bibliography I

Maddalena Bonanzinga.
On the Hausdorff Number of a Topological Space.
Houston Journal of Mathematics, 39(3):1013–1030, 2013.

Jean Goubault-Larrecq.
Non-Hausdorff Topologies and Domain Theory.
Cambridge University Press, 2013.

Anatoly Gryzlov.
Two theorems on the cardinality of topological spaces.
Soviet Math. Dokl., 21(2):506–509, 1980.

Istvan Juhasz.
Cardinal Functions in Topology - ten years later.
Mathematical Centre Tracts 123, Amsterdam, 1980.

Bibliography II

Petra Staynova.
A Note on the Hausdorff Number of Compact Spaces.
Proceedings of the 42 Spring Conference of the Union of Bulgarian Mathematicians, pages 248–253, 2013.