A Transcendental Meditation on Schanuel's Conjecture

Petra Staynova

University of Leicester

January 27, 2016

Not All Numbers Are Created Equal

We are all familiar with groups, rings, and fields of numbers

$$\mathbb{N}, \qquad \mathbb{Z}, \qquad \mathbb{Q}, \qquad \mathbb{R}, \qquad \mathbb{C}.$$

However, there is more than meets the eye in the last two cases. We are missing the algebraic numbers!

Definition (algebraic number)

An *algebraic number* is a number which is the zero of a polynomial with integer coefficients. We denote the set of algebraic numbers by $\overline{\mathbb{Q}}$.

For example

$$163 \;\; (x - 163 = 0), \quad \sqrt{2} \;\; (x^2 - 2 = 0), \quad i \;\; (x^2 + 1 = 0).$$

Some Numbers Have Dependencies...

Definition (linearly dependent)

Let $\alpha_1, \ldots, \alpha_n$ be real or complex numbers. We say they are *linearly dependent over* $\mathbb{Q}$ if and only if there is a linear function $p(x_1, \ldots, x_n) = \lambda_1 x_1 + \lambda_2 x_2 + \ldots \lambda_n x_n$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{Q}$ such that $p(\alpha_1, \ldots, \alpha_n) = 0$. Similarly, we may define linear dependence over $\mathbb{Z}$.

We say that the numbers $\alpha_1, \ldots, \alpha_n$ are linearly independent if they are not linearly dependent.

Definition (algebraically dependent)

Let $\alpha_1, \ldots, \alpha_n$ be real or complex numbers. We say they are *algebraically dependent over* $\mathbb{Q}$ if and only if there is a polynomial $p \in \mathbb{Q}[x_1, \ldots, x_n]$ with rational coefficients such that $p(\alpha_1, \ldots, \alpha_n) = 0$.

We say that the numbers $\alpha_1, \ldots, \alpha_n$ are algebraically independent if and only if they are not algebraically dependent.

Transcendental Numbers

Definition (transcendental number)

We call a number $\alpha \in \mathbb{C}$ *transcendental* if and only if it is not algebraic.

Note that there are countably many algebraic numbers. Thus, the majority of numbers 'out there' are transcendental. But, it is very difficult to find, or to demonstrate, even one transcendental number!

Theorem (Hermite's Theorem (1873))

Euler's constant, e, is transcendental.

Theorem (Hermite (1873), Lindemann (1882))

If α is a non-zero complex number, then at least one of α, e^{α} is transcendental.

Game Time: Transcendental or Algebraic?

- e (OK, we already saw that)
- π - transcendental, consequence of the Hermite-Lindemann Theorem
- $e^{\sqrt{2}}$ - transcendental, again consequence of the Hermite-Lindemann Theorem
- $\sum_{n=0}^{\infty} \frac{(6n)!n!}{(3n)!(2n)!^2}$ - algebraic (Beukers and Heckman, 1989)
- $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ - the Riemann (or Euler) zeta function, evaluated at 4
- $\sum_{n=1}^{\infty} \frac{(20n)!n!}{(10n)!(7n)!(4n)!}$ - algebraic (Beukers and Heckman, 1989)
- $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ - still unknown (Catalan's constant)
- $0.01101001\ldots$ (the Thue-Morse number) - transcendental
- $\log 2$ - transcendental by the Hermite-Lindemann Theorem
- the unique real root of $\cos x = x$ - transcendental, by the Hermite-Lindemann Theorem (Dottie Number)

The Lindemann-Weierstraß Theorem

Theorem (Lindemann-Weierstraß)

If $x_1, \ldots, x_n \in \overline{\mathbb{Q}}$ are $\mathbb{Q}$-linearly independent, then the numbers $e^{x_1}, \ldots, e^{x_n}$ are $\mathbb{Q}$-algebraically independent.

Proofs of the Lindemann-Weierstraß Theorem

- Lindemann approach
- Weierstraß approach
- Niven approach (Galois Theory)

The Gel'fond-Schneider Theorem and Baker's Theorem

Theorem (Gel'fond-Schneider)

If $\alpha, \beta \in \overline{\mathbb{Q}} \setminus \{0\}$, $\alpha \neq 1$, and $\beta \notin \mathbb{Q}$, then any value of α^β is transcendental.

Theorem (Baker's Theorem)

If $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ and $\log \alpha_1, \ldots, \log \alpha_n$ are $\mathbb{Q}$-linearly independent, then the numbers $1, \log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\overline{\mathbb{Q}}$.

The Six Exponentials Theorem

Theorem (Six Exponentials)

Let $x_1, x_2 \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$, and let $y_1, y_2, y_3 \in \mathbb{C}$ also be linearly independent over $\mathbb{Q}$. Then at least one of the six numbers

$$e^{y_1 x_1}, e^{y_1 x_2}, e^{y_2 x_1}, e^{y_2 x_2}, e^{y_3 x_1}, e^{y_3 x_2}$$

is transcendental (over $\mathbb{Q}$).

Note

- Special case attributed to Siegel in a paper by L. Alaoglu and P. Erdős in 1944.
- Two independent proofs of the Six Exponentials Theorem were published by S. Lang and K. Ramachandra.

...Wait! Wasn't There Some Conjecture Involved?

Conjecture (Schanuel)

If $\alpha_1, \ldots, \alpha_n$ are n linearly-independent over $\mathbb{Q}$ complex numbers, then at least n of the following $2n$ numbers are algebraically independent over $\mathbb{Q}$:

$$\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}.$$

Consequences of Schanuel's Conjecture Which are Conjectures

By induction on n, one can use Schanuel's Conjecture to obtain the algebraic independence of

$$e+\pi, e\pi, \pi^{e}, e^{e}, e^{e^{2}}, \ldots, e^{e^{e}}, \ldots, \pi^{\pi}, \pi^{\pi^{2}}, \ldots, \pi^{\pi^{\pi}}, \ldots$$

and of

$$\log \pi, \log(\log 2), \pi \log 2, (\log 2)(\log 3), 2^{\log 2}, (\log 2)^{\log 3}, \ldots.$$

Consequences of Schanuel's Conjecture Which are Conjectures (cont'd)

Conjecture

If $x_1, x_2 \in \mathbb{C}$ are $\mathbb{Q}$-linearly independent, then at least 2 of the 4 numbers $x_1, x_2, e^{x_1}, e^{x_2}$ are algebraically independent.

We obtain the algebraic independence of:

1. e and π;
2. e and e^e ;
3. π and e^π;
4. $\log 2$ and $\log 3$;
5. $\log 2$ and $2^{\log 2}$.

Consequences of Schanuel's Conjecture Which are Conjectures (cont'd)

To give an idea of the difficulty of these seeminly innocuous consequences, item 3 was not proven until 1996:

Theorem (Nesterenko)

π and e^{π} are algebraically independent.

The Four Exponentials Conjecture

We also don't know if there exist two logaritms of algebraic numbers which are algebraically independent.

Conjecture (Four Exponentials)

Given $\alpha_1, \ldots, \alpha_4 \in \mathbb{C}$ such that $(\log \alpha_1)(\log \alpha_4) = (\log \alpha_2)(\log \alpha_3)$, then either $\log \alpha_1$ and $\log \alpha_2$ are linearly dependent, or else $\log \alpha_1$ and $\log \alpha_3$ are linearly dependent.

Conjecture (Four Exponentials, restated)

If $\alpha_1, \alpha_2, \beta_2, \beta_2 \in \mathbb{C}$ are such that α_1, α_2 are linearly independent over $\mathbb{Q}$ and β_1, β_2 are $\mathbb{Q}$-linearly independent, then at least one of the four numbers

$$e^{\alpha_1\beta_1}, e^{\alpha_1\beta_2}, e^{\alpha_2\beta_1}, e^{\alpha_2\beta_2},$$

is transcendental.

Corollaries of Four Exponentials

Corollary

If for some $\alpha \in \mathbb{C}$*, both* $2^\alpha \in \mathbb{N}$ *and* $3^\alpha \in \mathbb{N}$ *, then* $\alpha \in \mathbb{N}$*.*

It is interesting to ask:

Open Question

If $3^\alpha - 2^\alpha \in \mathbb{N}$ for $\alpha \in \mathbb{C}$, can we deduce that either $\alpha \in \mathbb{N}$ or $\alpha \in \mathbb{C} \setminus \overline{\mathbb{Q}}$?

Proposition (PS)

Schanuel's Conjecture implies that if $3^\alpha - 2^\alpha \in \mathbb{N}$*, then* $\alpha \in \mathbb{Q}$ *or* $\alpha \in \mathbb{C} \setminus \overline{\mathbb{Q}}$*.*