Schanuel's Conjecture

Petra Staynova

Durham University

December 5, 2013

The Conjecture

Conjecture (Schanuel)

If $\alpha_1, \ldots, \alpha_n$ are n linearly-independent over $\mathbb{Q}$ complex numbers, then at least n of the following $2n$ numbers are algebraically independent over $\mathbb{Q}$:

$$\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}.$$

The Hermite-Lindemann Theorem

Theorem (Hermite-Lindemann)

If x is a non-zero complex number, then at least one of x, e^x is transcendental.

Proposition

The following numbers are transcendental:

1. e
2. π
3. $\log 2$
4. $e^{\sqrt{2}}$

The Lindemann-Weierstraß Theorem

Theorem (Lindemann-Weierstraß)

If $x_1, \dots, x_n \in \overline{\mathbb{Q}}$ *are* $\mathbb{Q}$*-linearly independent, then the numbers* $e^{x_1}, \dots, e^{x_n}$ *are* $\mathbb{Q}$*-algebraically independent.*

Note (Proofs of the Lindemann-Weierstraß Theorem)

- Lindemann approach
- Weierstraß approach
- Niven approach (Galois Theory)

The Gel'fond-Schneider Theorem and Baker's Theorem

Theorem (Gel'fond-Schneider)

If $\alpha, \beta \in \overline{\mathbb{Q}} \setminus \{0\}$, $\alpha \neq 1$, *and* $\beta \notin \mathbb{Q}$, *then any value of* α^{β} *is transcendental.*

Theorem (Baker's Theorem)

If $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ *and* $\log \alpha_1, \ldots, \log \alpha_n$ *are* $\mathbb{Q}$*-linearly independent, then the numbers* $1, \log \alpha_1, \ldots, \log \alpha_n$ *are linearly independent over* $\overline{\mathbb{Q}}$*.*

The Six Exponentials Theorem

Theorem (Six Exponentials)

Let $x_1, x_2 \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$, and let $y_1, y_2, y_3 \in \mathbb{C}$ also be linearly independent over $\mathbb{Q}$. Then at least one of the six numbers

$$e^{y_1 x_1}, e^{y_1 x_2}, e^{y_2 x_1}, e^{y_2 x_2}, e^{y_3 x_1}, e^{y_3 x_2}$$

is transcendental (over $\mathbb{Q}$).

Note

- Special case attributed to Siegel in a paper by L. Alaoglu and P. Erdős in 1944.
- Two independent proofs of the Six Exponentials Theorem were published by S. Lang and K. Ramachandra.
- Can also be deduced from a much more general result by Theodor Schneider.

Consequences of Schanuel's Conjecture Which are Conjectures

By induction on n, one can use Schanuel's Conjecture to obtain the algebraic independence of

$$e+\pi, e\pi, \pi^e, e^e, e^{e^2}, \dots, e^{e^e}, \dots, \pi^\pi, \pi^{\pi^2}, \dots, \pi^{\pi^\pi}, \dots$$

and of

$$\log\pi, \log(\log 2), \pi\log 2, (\log 2)(\log 3), 2^{\log 2}, (\log 2)^{\log 3}, \dots.$$

Consequences of Schanuel's Conjecture Which are Conjectures (cont'd)

Conjecture

If $x_1, x_2 \in \mathbb{C}$ are $\mathbb{Q}$-linearly independent, then at least 2 of the 4 numbers $x_1, x_2, e^{x_1}, e^{x_2}$ are algebraically independent.

We obtain the algebraic independence of:

1. e and π;
2. e and e^e ;
3. π and e^π;
4. $\log 2$ and $\log 3$;
5. $\log 2$ and $2^{\log 2}$.

Consequences of Schanuel's Conjecture Which are Conjectures (cont'd)

To give an idea of the difficulty of these seeminly innocuous consequences, item **??** was not proven until 1996:

Theorem (Nesterenko)

π and e^{π} are algebraically independent.

The Four Exponentials Conjecture

We also don't know if there exist two logaritms of algebraic numbers which are algebraically independent.

Conjecture (Four Exponentials)

Given $\alpha_1, \ldots, \alpha_4 \in \mathbb{C}$ such that $(\log \alpha_1)(\log \alpha_4) = (\log \alpha_2)(\log \alpha_3)$, then either $\log \alpha_1$ and $\log \alpha_2$ are linearly dependent, or else $\log \alpha_1$ and $\log \alpha_3$ are linearly dependent.

Conjecture (Four Exponentials, restated)

If $\alpha_1, \alpha_2, \beta_2, \beta_2 \in \mathbb{C}$ are such that α_1, α_2 are linearly independent over $\mathbb{Q}$ and β_1, β_2 are $\mathbb{Q}$-linearly independent, then at least one of the four numbers

$$e^{\alpha_1\beta_1}, e^{\alpha_1\beta_2}, e^{\alpha_2\beta_1}, e^{\alpha_2\beta_2},$$

is transcendental.

Corollaries of Four Exponentials

Corollary

If for some $\alpha \in \mathbb{C}$, both $2^\alpha \in \mathbb{N}$ and $3^\alpha \in \mathbb{N}$, then $\alpha \in \mathbb{N}$.

It is interesting to ask:

Open Question

If $3^\alpha - 2^\alpha \in \mathbb{N}$ for $\alpha \in \mathbb{C}$, can we deduce that either $\alpha \in \mathbb{N}$ or $\alpha \in \mathbb{C} \setminus \overline{\mathbb{Q}}$?

Proposition (PS)

Schanuel's Conjecture implies that if $3^\alpha - 2^\alpha \in \mathbb{N}$, then $\alpha \in \mathbb{Q}$ or $\alpha \in \mathbb{C} \setminus \overline{\mathbb{Q}}$.

Proof

Assume Schanuel's Conjecture and consider the set $\{\log 2, \log 3, \alpha \log 2, \alpha \log 3\}$ for α an irrational algebraic number. This set is $\mathbb{Q}$-linearly independent, so by SC,

$$\mathrm{trdeg}_{\mathbb{Q}}(\mathbb{Q}\left(\log 2, \log 3, \alpha \log 2, \alpha \log 3, 2, 3, 2^{\alpha}, 3^{\alpha}\right)) \geqslant 4.$$

Noting that

$$\mathbb{Q}\left(\log 2, \log 3, \alpha \log 2, \alpha \log 3, 2, 3, 2^{\alpha}, 3^{\alpha}\right) = \mathbb{Q}\left(\log 2, \log 3, 2^{\alpha}, 3^{\alpha}\right)$$

and applying properties of bases of extension fields, we have

$$\mathrm{trdeg}_{\mathbb{Q}}(\mathbb{Q}\left(\log 2, \log 3, 2^{\alpha}, 3^{\alpha}\right)\}) = 4.$$

Proof, cont'd

Hence, $3^\alpha - 2^\alpha$ is transcendental for α algebraic irrational.
By the contrapositive, we have that if $3^\alpha - 2^\alpha \in \mathbb{N}$, then α cannot be algebraic irrational, so $\alpha \in \mathbb{Q}$ or $\alpha \in \mathbb{C} \setminus \overline{\mathbb{Q}}$.

Consequences of Schanuel's Conjecture Which are Conjectures (cont'd)

Gel'fond (in 1948) and Schneider (in 1952) conjectured that:

Conjecture

If $\alpha, \beta \in \overline{\mathbb{Q}}$ and if β has degree $d \geqslant 2$, then $\operatorname{trdeg}_{\mathbb{Q}}\left(\mathbb{Q}\left(\alpha^{\beta}, \ldots, \alpha^{\beta^{d-1}}\right)\right) = d - 1$.

Conjecture (Gel'fond)

If $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ are linearly independent over $\mathbb{Q}$, and $\beta_1, \ldots, \beta_n \in \overline{\mathbb{Q}} \setminus \{0\}$ are such that $\log \beta_1, \ldots, \log \beta_n$ are also linearly independent over $\mathbb{Q}$, then

$$e^{\alpha_1}, \ldots, e^{\alpha_n}, \log \beta_1, \ldots, \log \beta_n$$

are $\overline{\mathbb{Q}}$-algebraically independent.

Even more conjectural consequences of Schanuel's Conjecture

Conjecture

[Algebraic Independence of Logarithms] Let $\beta_1, \ldots, \beta_n \in \overline{\mathbb{Q}} \setminus \{0\}$ and suppose that $\log \beta_1, \ldots, \log \beta_n$ are $\mathbb{Q}$-linearly independent. Then $\log \beta_1, \ldots, \log \beta_n$ are $\overline{\mathbb{Q}}$-algebraically independent.

Conjecture

If $\alpha, \beta_1, \ldots, \beta_n \in \overline{\mathbb{Q}}$, $\alpha \neq 0, 1$, and $1, \beta_1, \ldots, \beta_n$ are linearly independent over $\mathbb{Q}$, then $\log \alpha, \alpha^{\beta_1}, \ldots, \alpha^{\beta_n}$ are $\overline{\mathbb{Q}}$-algebraically independent.

Even more conjectural consequences of Schanuel's Conjecture

Lang and Ramachandra independently stated special cases of yet another conjecture which follows from Schanuel's Conjecture:

Conjecture (Lang and Ramachandra)

If $\alpha_1, \ldots, \alpha_n$ are $\mathbb{Q}$-linearly independent, and β is a transcendental number, then

$$\operatorname{trdeg}_{\mathbb{Q}}(\mathbb{Q}\left(e^{\alpha_1}, \ldots, e^{\alpha_n}, e^{\alpha_1 \beta}, \ldots, e^{\alpha_n \beta}\right)) \geqslant n-1.$$

An interesting consequence

Another interesting consequence is:

Conjecture

The numbers

$$e, e^{\pi}, e^{e}, e^{i}, \pi, \pi^{\pi}, \pi^{e}, \pi^{i}, 2^{\pi}, 2^{e}, 2^{i}, \log \pi, \log 2, \log 3, \log \log 2, (\log 2)^{\log 3}, 2^{\sqrt{2}}$$

are $\mathbb{Q}$-algebraically independent (and, in particular, they are transcendental).

Jump to Relations to Model Theory

Lang's Conjecture

We now turn to a conjecture by Lang.

Definition

We define the field E by transfinite induction on the ordinals:

1. $E_0 = \overline{\mathbb{Q}}$,
2. $E_{n+1} = \overline{E_n \left(e^x : x \in E_n\right)}$,
3. $E = E_\omega = \bigcup_{n \leqslant \omega} E_n$

Note

For ordinals $\alpha > \omega$, $E_\alpha = E$. In particular, $E_{\omega+1} = \overline{E_\omega \left(e^x : x \in E_\omega\right)} = \overline{E \left(e^x : x \in E\right)} = E$.

Lang's Conjecture

Proposition

Schanuel's Conjecture implies that $\pi \notin E$*.*

Definition

We define the field L by

1. $L_0 = \overline{\mathbb{Q}}$,
2. $L_{n+1} = \overline{L_n\left(\log x : x \in \mathbb{E}_n\right)}$,
3. $L = L_\omega = \bigcup_{n<\omega} L_n$,

again noting that $L_{\omega+1} = L$.

Lang's Conjecture

Definition (linearly disjoint field extensions)

Let $F \supset K$ be a field extension and $K \subseteq F_1, F_2 \subseteq F$ be two subextensions. We say they are *linearly disjoint over K* if and only if whenever $\{x_1, \ldots, x_n\} \subset F_1$ is linearly independent over K, then $\{x_1, \ldots, x_n\}$ is also linearly independent over F_2.

Theorem (Lang's Exercise)

Schanuel's Conjecture implies that the fields E and L are linearly disjoint over $\mathbb{Q}$.

Lang's Conjecture, corollaries

Corollary

Schanuel's Conjecture implies that:

1. $L \cap E = \overline{\mathbb{Q}}$;
2. $\pi \notin E$;

The following corollary to Lang is interesting in light of the previous Conjectures:

Corollary

Schanuel's Conjecture implies that:

1. $\pi, \log \pi, \log\log \pi, \ldots$ *are algebraically independent over* E;
2. $e, e^e, e^{e^e}, \ldots$ *are algebraically independent over* L;

Jump to Relations to Model Theory

Chow's Interesting Result I

We note that the Hermite-Lindemann Theorem can be restated as:

Theorem

The only solution to equation

$$e^{\alpha} = \beta$$

in the algebraic numbers is $\alpha = 0, \beta = 1$.

We know that the equation has many solutions for $\alpha, \beta \in \mathbb{C}$. But can we do better in narrowing down the domain over which it still has solutions? A natural idea would be to take $\overline{\mathbb{Q}}$ and close it with respect to taking exp and log, which leads us to the following definition:

Chow's Interesting Result II

Definition

A subfield F of $\mathbb{C}$ is *closed under* exp *and* log if (1) $\exp(x) \in F$ for all $x \in F$ and (2) $\log(x) \in F$ for all nonzero $x \in F$, where log is the branch of the natural logarithm function such that $-\pi < \mathrm{I}m(\log x) \leqslant \pi$ for all x. The *field* $\mathbb{E}$ *of EL numbers* is the intersection of all subfields of $\mathbb{C}$ that are closed under exp and log.

Now, let us make the question a bit more specific: rather than considering pairs (α, β), we consider the special case when $\alpha = -\beta$, so now we ask whether the equation

$$\alpha + e^{\alpha} = 0 \tag{1}$$

has a real root in $\mathbb{E}$. In [**?**], Timothy Chow claims that the Conjecture we have just stated is still unsolved:

Chow's Interesting Result III

Conjecture (Chow)

The real root R of $\alpha + e^{\alpha} = 0$ is not in $\mathbb{E}$.

Theorem

Schanuel's Conjecture implies that the real root R of $\alpha + e^{\alpha} = 0$ is not in $\mathbb{E}$.

In fact, Schanuel's Conjecture implies a stronger result, due to Lin [**?**]:

Theorem

Schanuel's Conjecture implies that whenever $f(x, y) \in \overline{\mathbb{Q}}[x, y]$ is an irreducible polynomial and $f(\alpha, \exp(\alpha)) = 0$ for some $\alpha \in \mathbb{C} \setminus \{0\}$, then $\alpha \notin \mathbb{L}$, where $\mathbb{L}$ is the smallest algebraically closed subfield of $\mathbb{C}$ that is closed under exp *and* log.

Jump to Relations to Model Theory

Even even more consequences! I

A curious result is given by Sondow:

Theorem

Assuming Schanuel's Conjecture, let $z, w \in \mathbb{C} \setminus \{0, 1\}$*. If both* $z^w, w^z \in \overline{\mathbb{Q}}$*, then* z *and* w *are either both rational or both transcendental.*

There is another very interesting consequence of Schanuel's Conjecture by Guiseppina Terzo, concerning algebraic relations among the elements of the exponential ring $(\mathbb{C}, e^x)$. Let us first give the formal definition, found in:

Definition

An *exponential ring* is a pair (R, E) with R a commutative ring with 1 and $E : R \to \mathcal{U}(R)$ a morphism of the additive group of R into the multiplicative group of units of R satisfying $E(x + y) = E(x).E(y)$ for all $x, y \in R$, and $E(0) = 1$.

Even even more consequences! II

So, intuitively, E plays the role of the exponential function in the commutative ring R. For her result, Terzo uses a more general version of Schanuel's Conjecture, which holds for any exponential ring:

Conjecture (Schanuel's Condition)

An exponential ring R satisfies Schanuel's Condition if R is a characteristic 0 domain and whenever $\alpha_1, \ldots, \alpha_n$ in R are linearly independent over $\mathbb{Q}$, the ring $\mathbb{Z}[\alpha_1, \ldots, \alpha_n, E(\alpha_1), \ldots, E(\alpha_n)]$ has transcendence degree at least n over $\mathbb{Q}$.

We recall that:

Definition

The characteristic of a field K is the smallest positive integer n with the property $nx = 0$ for all $x \in K$, and it is zero if no such n exists.

With these preliminaries in mind, Terzo's result states:

Even even more consequences! III

Theorem

Assuming Schanuel's Conjecture, there are no further relations between π *and* i *except the known ones,* $e^{i\pi} = -1$ *and* $i^2 = -1$.

Connections with Model Theory, take I

Definition (decidability)

A theory is *decidable* iff there is an effective procedure that, given an arbitrary formula expressible in the language of the theory, decides whether the formula is a member of the theory or not.

Open Question (Tarski, 1951)

Is the theory of the real field with exponentiation, $\mathbb{R}_{\text{exp}}$ decidable?

Theorem (McIntyre and Wilkie, 1996)

Schanuel's Conjecture implies that the real field with exponentiation, $\mathbb{R}_{\text{exp}}$*, is decidable.*

Zilber's Result

"It's always a pleasure to introduce ideas from model theory to people who do real mathematics." Professor Boris Zilber

Definition

Let $X \subseteq K$ be finite. We define a *dimension*

$$\partial(X) = \sup\{\mathrm{trdeg}(Y \cup E(\mathrm{span}(Y)) - \mathrm{lindim}(Y) : X \subseteq Y \text{ is finite}\}$$

and a *closure operator*

$$\mathrm{cl}(X) = \{a : \partial(X) = \partial(Xa)\}.$$

Zilber's Result

Theorem (Zilber, 2005)

For all uncountable cardinals κ, there is a unique model of Φ of cardinality κ. If $(K, +, ., E) \models \Phi$, then every definable subset of K is countable or with countable complement. If $A \subseteq K$ is finite and $a, b \notin cl(A)$ there is an automorphism of K taking a to b.

Zilber's Result

Moreover, if $(K, +, ., E) \vDash \Phi$, then $(K, +, ., E)$ satisfies the following five axioms:

Axiom (EXP)

$$E(x_1 + x_2) = E(x_1).E(x_2)$$
$$\ker(E) = \pi\mathbb{Z}, \text{ some } \pi \in K.$$

Axiom (SCH)

$$\mathrm{trdeg}(X \cup E(X)) - \mathrm{lindim}(X) \geqslant 0,$$

Zilber's Result

Axiom (EC)

For any non-overdetermined irreducible system of polynomial equations

$$P(x_1, \dots, x_n, y_1, \dots, y_n) = 0$$

there exists a generic solution satisfying

$$y_i = E(x_i) \ i = 1, \dots, n.$$

Axiom (CC)

Analytic subsets of K^n of dimension 0 are countable.

Axiom (ACF_0)

Axioms for algebraically closed fields of characteristic 0.

Even more from Zilber

Conjecture

The field of complex numbers with exponentiation, $\mathbb{C}_{\text{exp}}$, is isomorphic to the unique field with exponentiation K_E of cardinality $2^{\aleph_0}$.

We conclude with a final interesting result from Model Theory which runs in a similar vein:

Theorem

There are at most countably many essential counterexamples to Schanuel's Conjecture.

Want more fun consequences?

- Lang's Exercise
- Chow's Interesting Result
- Terso's Curious Consequence
- Some of the Proofs we have omitted