Some open questions concerning Lindelöf-type properties

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Introduction

Note 1

All spaces are assumed Hausdorff.

Definition 2

X is *weakly Lindelöf* if for every open cover $\mathcal{U}$ of X there is a countable subfamily $\mathcal{U}' \subseteq \mathcal{U}$ such that $X = \overline{\bigcup \mathcal{U}'}$.

Definition 3

We define the *weakly Lindelöf number* of a set X as

$$\mathrm{wL}(X) =$$

$$\omega.\min\{\kappa : \forall \mathcal{U} \text{ open cover of } X \exists \mathcal{U}' \subseteq \mathcal{U}, |\mathcal{U}'| \leqslant \kappa : X = \overline{\bigcup \mathcal{U}'}\}$$

Definition 4

X is called *hereditarily weakly Lindelöf* if every subset of X is weakly Lindelöf.

In a manner similar to that of Definition 3, we can define a hereditarily weakly Lindelöf number.

Definition 5

$$\mathrm{hwL}(X) =$$
$$\omega.\min\{\kappa : \forall Y \subseteq X \forall \mathcal{U} \text{ open in } X \text{ cover of } Y$$
$$\exists \mathcal{U}' \subseteq \mathcal{U}, |\mathcal{U}'| \leqslant \kappa : Y = \overline{\bigcup \mathcal{U}'}\}$$

Definition 6

We call a space X *quasi-Lindelöf* if for every closed subset Y of X and every collection $\mathcal{U}$ of open in X sets such that $Y \subseteq \bigcup \mathcal{U}$, there is a countable subfamily $\mathcal{U}' \subseteq \mathcal{U}$ such that $Y \subset \overline{\bigcup \mathcal{U}'}$.

In other words, the property of being weakly Lindelöf is inherited only by closed subspaces.

Definition 7

We define the *quasi-Lindelöf number* of X as

$$\mathrm{qL}(X) =$$

$$\omega . \min\{\kappa : \forall Y \text{ closed } \subseteq X \forall \mathcal{U} \text{ open in } X \text{ cover of } Y$$

$$\exists \mathcal{U}' \subseteq \mathcal{U}, |\mathcal{U}'| \leqslant \kappa : Y = \overline{\bigcup \mathcal{U}'}\}$$

Fact 8

We (obviously) have

$$wL(X) \leqslant qL(X) \leqslant hwL(X) \leqslant hL(X)$$

Fact 9

$$wL(X) \leqslant qL(X) \leqslant L(X)$$

Fact 10

If X is normal, then $qL(X)$=$wL(X)$.

Open Question 11

Find an example of a completely regular weakly Lindelöf space that is not quasi-Lindelöf.

Fact 12

Bell, Ginsburgh, and Woods proved that every normal first countable weakly Lindelöf space has cardinality at most continuum.

Fact 13

Arhangelskii proved that every regular first countable quasi-Lindelöf space has cardinality at most continuum.

Open Question 14 (Bell, Ginsburgh, Woods)

Is Fact 12 is true for completely regular spaces?

Note that Fact 12 is not true for regular spaces: Bell, Ginsburgh and Woods have provided such an example.

Open Question 15 (Arhangelskii)

Is Fact 13 true for Hausdorff spaces?

Note 16

From now on, all spaces are assumed regular (T_3).

Definition 17 (Tall)

A space X is called *productively Lindelöf* if for every Lindelöf space Y, we have that $X \times Y$ is Lindelöf.

Fact 18

Compact and σ-compact spaces are productively Lindelöf.

Fact 19 (Hajnal, Juhasz)

There is a Lindelöf space that is not productively Lindelöf, i.e. Lindelöf spaces are not productively Lindelöf.

Definition 20 (PS)

We say that X is productively weakly Lindelöf if $X \times Y$ is weakly Lindelöf for every weakly Lindelöf space Y.

Definition 21 (PS)

We say that X is productively quasi-Lindelöf if $X \times Y$ is quasi-Lindelöf for every quasi-Lindelöf space Y.

Fact 22

Compact spaces are productively weakly Lindelöf.

Open Question 23 (PS)

Are compact spaces productively quasi-Lindelöf?

Definition 24

A space X is a *D-space* if whenever $f : X \to \tau$ is a neighborhood assignment (i.e. $x \in f(x)$) there is a closed discrete set D such that $\{f(x) : x \in D\}$ covers X.

Open Question 25 (van Dowen)

Is every Lindelöf space a D-space?

Definition 26 (PS)

A space X is a weakly D-space if whenever $f : X \to \tau$ is a neighborhood assignment, there is a closed discrete D such that $\{f(x) : x \in D\}$ is dense in X.

Open Question 27 (PS)

Is every Lindelöf space a weakly D-space?

Open Question 28 (PS)

Is every weakly-Lindelöf space a weakly D-space?

Definition 29

A *Michael space* is a Lindelöf space X such that $X \times \mathbb{I}$ is *not* Lindelöf.

Open Question 30 (Michael)

Is there a Michael space?

Definition 31 (PS)

A weakly Lindelöf space X is called *weakly Michael* if $X \times \mathbb{I}$ is not weakly Lindelöf.

Open Question 32 (PS)

Is there a weakly Michael space?

Definition 33 (PS)

A Lindelöf space X is called *strongly Michael* if $X \times \mathbb{I}$ is not weakly Lindelöf.

Open Question 34 (PS)

Is there a strongly Michael space?