Compactness-like Covering Properties

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Something to think about...

[Look at the board]

Introduction

One of the main generalisations of compactness is the notion of an H-closed space:

Definition (H-closed, general)

A topological space X is said to be *H-closed* iff it is closed in every Hausdorff space containing it as a subspace.

A more useful definition is:

Definition (H-closed)

A topological space X is said to be *H-closed* iff every open cover has a finite subfamily with dense union.

Countable generalisations

Another generalisation of compactness is the well-known Lindelöf property:

Definition (Lindelöf)

A topological space X is called *Lindelöf* iff every open cover has a countable subcover.

Example (Lindelöf spaces)

The following spaces are Lindelöf:

- Any countable topological space;
- Any space with the co-countable topology;
- A countable union of compact spaces;
- $\mathbb{R}$, with the Euclidean topology.
- The Sorgenfrey line $\mathbb{S}$ ($\mathbb{R}$ with the topology generated by the base $\mathcal{B} = \{[a, b) : a < b\}$).

Countable generalisations, cont'd

In 1959, Zdenek Frolik introduced a notion that combines the Lindelöf and H-closed properties:

Definition (weakly Lindelöf , [Fro59])

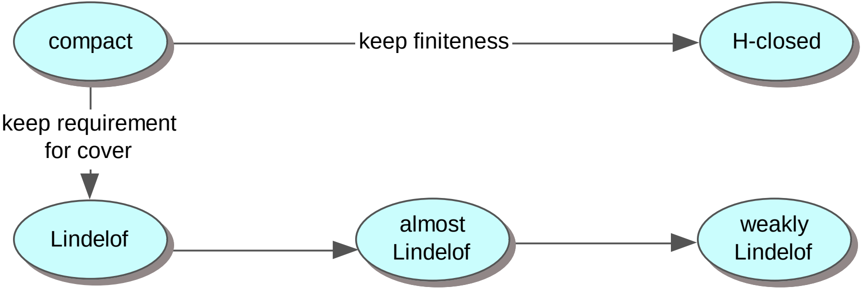
A topological space X is *weakly Lindelöf* if for every open cover $\mathcal{U}$ of X there is a countable subfamily $\mathcal{U}' \subseteq \mathcal{U}$ such that $X = \overline{\bigcup \mathcal{U}'}$.

Later on, while studying cardinal invariants, Dissanayeke and Willard introduced another generalisation of the Lindelöf property, which is stronger than the weakly Lindelöf one:

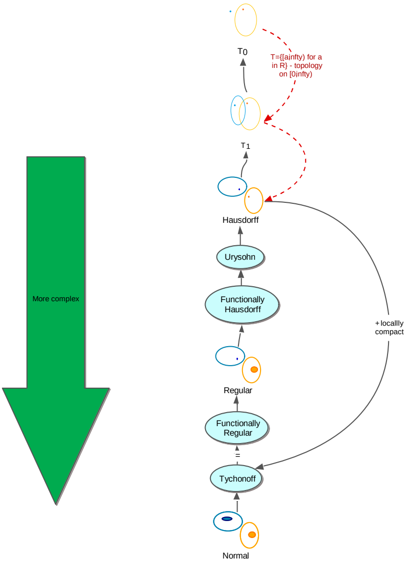
Definition (almost Lindelöf , [WD84])

A topological space X is *almost Lindelöf* iff for every open cover $\mathcal{U}$ of X there is a countable subfamily $\mathcal{U}' \subseteq \mathcal{U}$ such that $X = \bigcup \{\overline{U} : U \in \mathcal{U}'\}$.

Generalisations of compactness: a diagram



Separation Axioms



Properties of the generalisations

Proposition

- *Every regular Hausdorff H-closed space is compact.*
- *Every regular Lindelöf space is normal.*

Proposition

A regular almost Lindelöf space is Lindelöf.

Proposition

A normal weakly Lindelöf space is almost Lindelöf.

Properties of the generalisations

Proposition

The continuous image of an almost Lindelöf (weakly Lindelöf) space is almost Lindelöf (weakly Lindelöf).

Proposition

A clopen subset of an almost Lindelöf (weakly Lindelöf) space is almost Lindelöf (weakly Lindelöf).

Proposition

If X is almost Lindelöf (weakly Lindelöf) and Y is compact, then $X \times Y$ is almost Lindelöf (weakly Lindelöf).

Inheritance

Property	Compact	H-closed	Lindelof	Weakly Lindelof
Inherited by closed subspaces?	Yes	No (regularly closed)	Yes	No (regularly closed)

A modification

Definition (quasi-Lindelöf , [Arh79])

We call a space X *quasi-Lindelöf* if for every closed subset Y of X and every collection $\mathcal{U}$ of open in X sets such that $Y \subseteq \bigcup \mathcal{U}$, there is a countable subfamily $\mathcal{U}' \subseteq \mathcal{U}$ such that $Y \subset \overline{\bigcup \mathcal{U}'}$.

Proposition ([Sta12])

Let X be a topological space. The following are equivalent:

1. *X is quasi-Lindelöf.*
2. *Let $\mathcal{B}$ be a fixed base for X. Then for any closed subset $C \subset X$ and any cover $\mathcal{U}$ of C with $\mathcal{U} \subset \mathcal{B}$ there is a countable subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $C \subset \overline{\bigcup \mathcal{U}'}$.*

weakly Lindelöf not quasi-Lindelöf example

We modify an example from Song and Zhang [SZ10] and use ideas from Mysior [Mys81] to get:

Example

There exists a Urysohn weakly Lindelöf not quasi-Lindelöf space.

(the original example was of a Urysohn almost Lindelöf space which is not Lindelöf)

Weakly Lindelöf not quasi-Lindelöf example

Construction

Let $A = \{(a_\alpha, -1) : \alpha < \omega_1\}$ be an ω_1-long sequence in the set $\{(x, -1) : x \geqslant 0\} \subseteq \mathbb{R}^2$. Let $Y = \{(a_\alpha, n) : \alpha < \omega_1, n \in \omega\}$. Let $a = (-1, -1)$. Finally let $X = Y \cup A \cup \{a\}$.
We topologize X as follows:

- all points in Y are isolated;
- for $\alpha < \omega_1$ the basic neighborhoods of $(a_\alpha, -1)$ will be of the form

$$U_n(a_\alpha, -1) = \{(a_\alpha, -1)\} \cup \{(a_\alpha, m) : m \geqslant n\} \text{ for } n \in \omega$$

- the basic neighborhoods of $a = (-1, -1)$ are of the form

$$U_\alpha(a) = \{a\} \cup \{(a_\beta, n) : \beta > \alpha, n \in \omega\} \text{ for } \alpha < \omega_1.$$

Weakly Lindelöf not quasi-Lindelöf example

Let us point out that

Claim

The subset $A \subset X$ is closed and discrete in this topology.

Indeed, for any point $x \in X$ there is a basic neighborhood $U(x)$ such that $A \cap U(x)$ contains at most one point and also that $X \setminus A = \{a\} \cup Y$ is open (because $U_\alpha(a) \subset Y \cup \{a\}$). Hence X contains an uncontable closed discrete subset and therefore it cannot be Lindelöf.
Also,

Note

Note that for any open $U \ni a$ the set $X \setminus \overline{U}$ is at most countable.

Indeed, for any $\alpha < \omega_1$, $\overline{U_\alpha(a)} = U_\alpha(a) \cup \{(a_\beta, -1) : \beta > \alpha\}$. Hence $X \setminus \overline{U_\alpha(a)}$ is at most countable.
It is easily seen that X is Hausdorff. With a bit more effort, it can also be proven that X is Urysohn.

Weakly Lindelöf not quasi-Lindelöf example

Claim

The space X is weakly Lindelöf.

Let $\mathcal{U}$ be an open cover of X. Then there exists a $U(a) \in \mathcal{U}$ such that $a \in U(a)$. We can find a basic neighborhood $U_\beta(a) \subset U(a)$. Then $\overline{U_\beta(a)} \subset \overline{U(a)}$ and hence $X \setminus \overline{U(a)}$ will also be at most countable. Hence $X \setminus \overline{U(a)}$ can be covered by (at most) countably many elements of $\mathcal{U}$, say $\mathcal{U}^*$. Set $\mathcal{U}' = \mathcal{U}^* \cup \{U(a)\}$. Then, $X \subseteq \bigcup_{U \in \mathcal{U}'} \overline{U} \subseteq \overline{\bigcup_{U \in \mathcal{U}'} U}$. Therefore, X is weakly Lindelöf.

Note

In fact, this shows that X is even almost Lindelöf.

Weakly Lindelöf not quasi-Lindelöf example

Claim

The space X is not quasi-Lindelöf.

Consider the 1-neighborhood of a:

$$U_1(a) = \{a\} \cup \{(a_\beta, n) : \omega_1 > \beta > 1, n \in \omega\}.$$

We have that $C = X \setminus U_1(a)$ is closed.
We show the uncountable family of basic open sets $\mathcal{U} = \{U_0(a_\alpha, -1) : \alpha < \omega_1\}$ forms an open cover of C which has no countable subcover with dense union.
Note that the sets $U_0(a_\alpha, -1)$ are closed and open. Indeed, $X \setminus U_0(a_\alpha, -1) = \bigcup\{U_0(a_\beta, -1) : \omega_1 > \beta \neq \alpha\} \cup U_{\alpha+1}(a)$. Hence, if we remove even one of the $U_0(a_\alpha, -1)$, the point $(a_\alpha, -1)$ would remain uncovered. Therefore, X is not quasi-Lindelöf.

Another weakly Lindelöf not quasi-Lindelöf example

A shorter example shows that even regularity is not strong enough to make a weakly Lindelöf space quasi-Lindelöf :

Example

The Sorgenfrey plane is weakly Lindelöf but it is not quasi-Lindelöf.

The quasi-Lindelof property

Theorem

Every separable topological space X is quasi-Lindelöf.

Definition (ccc)

A topological space X satisfies the *countable chain condition* if every family of non-empty disjoint open subsets of X is countable.

Arhangelskii stated without proof that:

Theorem ([Arh79])

Every ccc space is weakly-Lindelöf.

In my 3d year project, I proved that:

Theorem ([Sta11])

Every ccc space is quasi-Lindelöf.

ccc implies quasi-Lindelöf - the proof

Suppose that X is CCC but not quasi-Lindelöf. Then there exists a closed nonempty set $F \subset X$ and an uncountable family $\Gamma = \{U_\alpha : \alpha < \beta\}$ ($\beta \geqslant \omega_1$) of non-empty open in X sets such that $F \subset \bigcup_{\alpha<\beta} U_\alpha$, but for any countable $\Gamma' \subset \Gamma$ we have that $F \setminus \overline{\bigcup \Gamma'} \neq \emptyset$.
We will construct an uncountable collection of nonempty disjoint open in X sets $\{V_\gamma : \gamma < \omega_1\}$, thus contradicting CCC.
Let $V_0 = U_0$. Then $F \setminus \overline{U_0} \neq \emptyset$ (and $X \setminus \overline{U_0} \neq \emptyset$). Hence

$$\emptyset \neq F \setminus \overline{U_0} \subset \bigcup \{U_\alpha : \alpha < \beta, \alpha > 0\}$$

and therefore

$$\emptyset \neq F \setminus \overline{U_0} = \bigcup \{U_\alpha \cap (F \setminus \overline{U_0}) : \alpha < \beta, \alpha > 0\}.$$

ccc implies quasi-Lindelöf - the proof cont'd

Thus we will have $\alpha_1 \geqslant 1$ such that $U_{\alpha_1} \cap (F \setminus \overline{U_0}) \neq \emptyset$ and moreover $U_{\alpha_1} \cap (X \setminus \overline{U_0}) \neq \emptyset$.
Let $V_1 = U_{\alpha_1} \cap (X \setminus \overline{U_0})$. Then $V_1 \neq \emptyset$, V_1 is open in X and $V_1 \cap V_0 = \emptyset$.
Again we have

$$\emptyset \neq F \setminus \overline{U_0 \cup U_{\alpha_1}} = \bigcup \{ U_\alpha \cap (F \setminus \overline{U_0 \cup U_{\alpha_1}}); \alpha < \beta, \alpha \notin \{0, \alpha_1\}\}.$$

Hence there is $U_{\alpha_2} \in \Gamma$, $\alpha_1 \notin \{0, \alpha_1\}$ with $U_{\alpha_2} \cap (F \setminus \overline{U_0 \cup U_{\alpha_1}}) \neq \emptyset$ and moreover $U_{\alpha_2} \cap (X \setminus \overline{U_0 \cup U_{\alpha_1}}) \neq \emptyset$.
Define $V_2 = U_{\alpha_2} \cap (X \setminus \overline{U_0 \cup U_{\alpha_1}})$. Then $V_2 \neq \emptyset$, V_2 is open in X and V_2 is disjoint from V_0, V_1. Let $\gamma_0 < \omega_1$ and suppose that we have already constructed a family $\{V_\delta : \delta < \gamma_0\}$ of non-empty, disjoint open in X sets with $V_\delta = U_{\alpha_\delta} \cap (X \setminus \overline{\cup \{U_{\alpha_\sigma} : \sigma < \delta\}})$, where $\alpha_\delta \notin \{\alpha_\sigma : \sigma < \delta\}$.

ccc implies quasi-Lindelöf - the proof cont'd

Since γ_0 is a countable ordinal and F is not weakly Lindelöf in X, we have that

$$\begin{aligned}\emptyset &\neq F \setminus \overline{\cup\{U_{\alpha_\delta} : \delta < \gamma_0\}} \\ &= \bigcup\{U_\alpha \cap (F \setminus \overline{\cup\{U_{\alpha_\delta} : \delta < \gamma_0\}}) : \alpha < \delta, \alpha \notin \{\alpha_\delta : \delta < \gamma_0\}\}.\end{aligned}$$

Hence we can choose α_{γ_0} such that

$$U_{\alpha_{\gamma_0}} \cap (F \setminus \overline{\cup\{U_{\alpha_\delta} : \delta < \gamma_0\}}) \neq \emptyset$$

and $\alpha_{\gamma_0} \notin \{\alpha_\delta : \delta < \gamma_0\}$. Hence moreover

$$U_{\alpha_{\gamma_0}} \cap (X \setminus \overline{\cup\{U_{\alpha_\delta} : \delta < \gamma_0\}}) \neq \emptyset.$$

ccc implies quasi-Lindelöf - the proof cont'd

Define $V_{\gamma_0} = U_{\alpha_{\gamma_0}} \cap (X \setminus \overline{\cup\{U_{\alpha_\delta} : \delta < \gamma_0\}})$.
Then $V_{\gamma_0} \neq \emptyset$, V_{γ_0} is open in X and by construction $V_{\gamma_0} \cap V_\delta = \emptyset$ for every $\delta < \gamma_0$.
Thus we have constructed a family $\{V_\gamma : \gamma < \omega_1\}$ of nonempty disjoint open in X sets, contradicting CCC.
Hence, X is quasi-Lindelöf.

What about the converse?

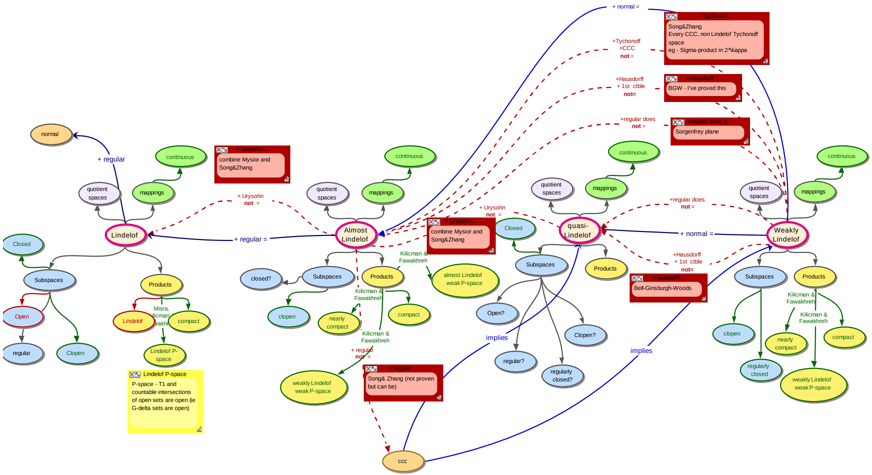
Though we have that ccc implies quasi-Lindelöf, the converse is false:

Example

The lexicographic square is quasi-Lindelöf, but is not ccc.

The quasi-Lindelöf property follows from the fact that the lexicographic square is compact, that it is not ccc can be found in [SS96].

Relations between Lindelöf-type covering properties



Open questions - preliminaries

The famous Tychonoff's Theorem states that an arbitrary product of compact spaces is compact. However, for Lindelöf spaces, this fails even in the finite case, as exemplified by the Sorgenfrey plane.
In [TAAJ11] Frank Tall introduced the productively Lindelöf property:

Definition (productively Lindelöf)

A space X is called *productively Lindelöf* if for every Lindelöf space Y, the product $X \times Y$ is Lindelöf.

It is well-known that compact spaces are productively Lindelöf; Tall proved that under certain additional axioms, there exist other productively Lindelöf properties.

Open questions - preliminaries

Similarly, we can define productively weakly-Lindelöf:

Definition ((PS) productively weakly Lindelöf)

A space X is called *productively weakly Lindelöf* if for every weakly Lindelöf space Y, the product $X \times Y$ is weakly Lindelöf.

Proposition

If X is weakly Lindelöf and Y is compact, then $X \times Y$ is weakly Lindelöf.

This was stated by Song and Zhang in [SZ10], a proof can be found in [Sta11].

Open questions - continued

We can also define:

Definition ((PS) productively quasi-Lindelöf)

A space X is called *productively quasi-Lindelöf* if for every quasi-Lindelöf space Y, the product $X \times Y$ is quasi-Lindelöf.

Thus, it is natural to ask:

Open Question ([Sta12])

Are compact spaces productively quasi-Lindelöf?

This question is interesting even in the partial case:

Open Question ([Sta12])

Is the product of the unit interval $[0, 1]$ with a quasi-Lindelöf space, quasi-Lindelöf?

Open questions, cont'd

We know that there is a regular weakly Lindelöf space that is not quasi-Lindelöf, and also that every normal weakly Lindelöf space is quasi-Lindelöf. Hence, it makes sense to ask:

Open Question

In completely regular spaces, do the weakly Lindelöf and quasi-Lindelöf properties coincide?

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